

Question 3

Consider $\sum_{n=0}^{\infty} \frac{1}{2^n}$.

- (a) Identify the geometric data and derive the N th partial sum.
- (b) Find the sum and exact remainder.
- (c) Interpret the result as repeatedly adding half of the remaining distance to 2.

Question 3 – Solution**Step 1: Identify the series.**

The expansion is $1 + 1/2 + 1/4 + \cdots$, so the first term is $a = 1$ and the common ratio is $r = 1/2$. Since $|r| < 1$, the series converges.

Step 2: Derive the finite partial sum.

With indices 0 through N , there are $N + 1$ terms. The finite geometric formula gives

$$s_N = \frac{1 - (1/2)^{N+1}}{1 - 1/2} = 2 - 2^{-N}.$$

Because $2^{-N} \rightarrow 0$,

$$S = \lim_{N \rightarrow \infty} s_N = 2.$$

Step 3: Compute and interpret the remainder.

The exact gap is $R_N = S - s_N = 2^{-N}$. Starting at $s_0 = 1$, each new term equals one-half of the current distance to 2; the gap is halved at every step. The process never exceeds 2, but it can get arbitrarily close.