

Question 2

Consider $\sum_{n=1}^{\infty} \frac{1}{n^4}$.

- (a) Prove convergence and compare its speed with the Basel series.
- (b) State Euler's classical value $\zeta(4)$.
- (c) Using that value, compute the even-indexed and odd-indexed fourth-power subseries separately.

Question 2 – Solution**Step 1: Establish convergence.**

The series is a p -series with $p = 4 > 1$, so it converges. Moreover, $1/n^4 \leq 1/n^2$ for $n \geq 1$, so its tail decays substantially faster than the Basel-series tail.

Step 2: State the special value accurately.

Euler's classical evaluation is

$$\zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90} \approx 1.082323.$$

As with $\zeta(2)$, the p -series test proves convergence but not this exact constant.

Step 3: Extract the even contribution.

For even indices $n = 2k$,

$$\sum_{k=1}^{\infty} \frac{1}{(2k)^4} = \frac{1}{16} \sum_{k=1}^{\infty} \frac{1}{k^4} = \frac{\pi^4}{1440}.$$

Step 4: Extract the odd contribution.

Subtracting the even terms from the full series gives

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^4} = \frac{\pi^4}{90} - \frac{\pi^4}{1440} = \frac{\pi^4}{96}.$$

The even and odd pieces recombine to $\pi^4/90$, providing an algebraic check.