

Question 1

Consider the Basel series $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

- (a) Prove that the series converges using the p -series test.
- (b) State Euler's classical evaluation of its exact sum and give a decimal approximation.
- (c) Use an integral estimate to bound the remainder after N terms, and explain why convergence alone does not produce the exact constant.

Question 1 – Solution**Step 1: Classify the series.**

This is a p -series $\sum 1/n^p$ with $p = 2 > 1$. Therefore it converges. This test establishes existence of a finite sum but does not determine that sum.

Step 2: State the classical special value.

Euler's solution of the Basel problem gives the non-elementary identity

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \zeta(2) = \frac{\pi^2}{6} \approx 1.644934.$$

The equality with $\pi^2/6$ requires additional machinery, such as Euler's product for $\sin x$ or Fourier series; it does not follow from the p -series test alone.

Step 3: Quantify convergence.

Because $f(x) = 1/x^2$ is positive and decreasing,

$$\int_{N+1}^{\infty} \frac{dx}{x^2} \leq R_N \leq \int_N^{\infty} \frac{dx}{x^2},$$

so

$$\frac{1}{N+1} \leq R_N \leq \frac{1}{N}.$$

Thus the partial sums approach $\pi^2/6$ from below, with an error of order $1/N$.