

**Question 10**

Determine whether  $\sum_{n=1}^{\infty} \frac{1}{n^2 + \sin n}$  converges or diverges.

- (a) Show that the denominator is positive for every  $n \geq 1$ .
- (b) Use  $-1 \leq \sin n \leq 1$  to produce a simple upper comparison for the tail.
- (c) Confirm the classification by limit comparison with  $1/n^2$ .

**Question 10 – Solution****Step 1: Check positivity and the domain.**

For  $n = 1$ ,  $1 + \sin 1 > 0$ . For  $n \geq 2$ , use  $\sin n \geq -1$  to obtain

$$n^2 + \sin n \geq n^2 - 1 > 0,$$

so every term is defined and positive.

**Step 2: Create an upper bound for the tail.**

For  $n \geq 2$ ,

$$n^2 - 1 \geq \frac{3}{4}n^2,$$

because  $n^2 \geq 4$ . Hence

$$0 < \frac{1}{n^2 + \sin n} \leq \frac{1}{n^2 - 1} \leq \frac{4}{3n^2}.$$

The comparison series is a constant multiple of the convergent  $p$ -series  $\sum 1/n^2$ . The Direct Comparison Test therefore proves convergence of the tail, and adding the finite first term cannot change that conclusion.

**Step 3: Confirm the dominant behavior.**

Limit comparison gives the sharper asymptotic statement

$$\lim_{n \rightarrow \infty} \frac{1/(n^2 + \sin n)}{1/n^2} = \lim_{n \rightarrow \infty} \frac{n^2}{n^2 + \sin n} = 1,$$

because dividing numerator and denominator by  $n^2$  produces  $1/(1 + \sin n/n^2)$ , and  $|\sin n|/n^2 \leq 1/n^2 \rightarrow 0$ . The ratio limit 1 confirms that the oscillation in the denominator is asymptotically negligible.