

Question 9

Determine whether $\sum_{n=1}^{\infty} \frac{n}{n^2 + 1}$ converges or diverges.

- (a) Verify the nth-term condition.
- (b) Perform a limit comparison with the harmonic series.
- (c) Give a direct lower comparison that independently proves divergence.

Question 9 – Solution

Step 1: Check the terms and select a benchmark.

Dividing by n^2 shows $n/(n^2 + 1) = (1/n)/(1 + 1/n^2) \rightarrow 0$, so the n th-term test is inconclusive. The dominant-power behavior suggests the harmonic benchmark $b_n = 1/n$.

Step 2: Perform limit comparison.

$$\lim_{n \rightarrow \infty} \frac{n/(n^2 + 1)}{1/n} = \lim_{n \rightarrow \infty} \frac{n^2}{n^2 + 1} = 1.$$

The limit equals 1, which is finite and strictly positive. Since both series have positive terms and $\sum 1/n$ diverges, the Limit Comparison Test proves that the given series diverges.

Step 3: Verify with direct comparison.

For $n \geq 1$, $n^2 + 1 \leq 2n^2$, so taking positive reciprocals and multiplying by n gives

$$\frac{n}{n^2 + 1} \geq \frac{n}{2n^2} = \frac{1}{2n}.$$

The smaller series $\sum 1/(2n)$ diverges; therefore the larger given positive-term series also diverges. This direction of comparison is essential when proving divergence.