

Question 8

Determine whether $\sum_{n=1}^{\infty} \left(1 - \cos \frac{1}{n}\right)$ converges or diverges.

- (a) Verify that the terms approach zero.
- (b) Use $0 \leq 1 - \cos x \leq x^2/2$ to obtain a direct comparison.
- (c) Confirm the quadratic scale by computing a limit-comparison ratio.

Question 8 – Solution**Step 1: Check the terms.**

Since $1/n \rightarrow 0$ and cosine is continuous, $1 - \cos(1/n) \rightarrow 1 - \cos 0 = 0$. The necessary condition holds, so a convergence test is still needed.

Step 2: Build a direct comparison.

The standard small-angle inequality

$$0 \leq 1 - \cos x \leq \frac{x^2}{2}$$

holds for every real x . Substituting $x = 1/n$ yields

$$0 \leq 1 - \cos(1/n) \leq \frac{1}{2n^2}.$$

Both sides are nonnegative, and $\sum 1/(2n^2)$ is a convergent p -series with $p = 2 > 1$. The Direct Comparison Test therefore proves convergence.

Step 3: Confirm the asymptotic scale.

The standard Taylor limit confirms more precisely that

$$\lim_{n \rightarrow \infty} \frac{1 - \cos(1/n)}{1/n^2} = \lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2} = \frac{1}{2}.$$

Thus the summand is asymptotic to $1/(2n^2)$. The finite positive ratio also satisfies the hypotheses of the Limit Comparison Test, independently confirming the conclusion.