

Question 6

Determine whether $\sum_{n=1}^{\infty} e^{-\sqrt{n}}$ converges or diverges.

- (a) Verify the necessary nth-term condition.
- (b) Group indices into blocks $k^2 \leq n < (k+1)^2$ and bound the sum of each block from above.
- (c) Compare the block bounds with a convergent series.

Question 6 – Solution

Step 1: Check the necessary condition.

Let $a_n = e^{-\sqrt{n}}$. Because $\sqrt{n} \rightarrow \infty$, the exponent tends to $-\infty$ and $a_n \rightarrow 0$. The nth-term condition holds but does not prove convergence.

Step 2: Organize the tail into square blocks.

In the block $k^2 \leq n < (k+1)^2$, there are exactly $2k+1$ indices. Also $\sqrt{n} \geq k$, so the decreasing exponential satisfies $e^{-\sqrt{n}} \leq e^{-k}$. Therefore

$$\sum_{n=k^2}^{(k+1)^2-1} e^{-\sqrt{n}} \leq (2k+1)e^{-k}.$$

Step 3: Classify the series of block bounds.

It remains to check $\sum_{k=1}^{\infty} (2k+1)e^{-k}$. Let $q = e^{-1} < 1$. Then

$$2 \sum_{k=1}^{\infty} kq^k + \sum_{k=1}^{\infty} q^k,$$

Both series converge: $\sum q^k$ is geometric, and $\sum kq^k = q/(1-q)^2$ follows by differentiating the geometric series.

Step 4: Conclude by comparison.

The original nonnegative series is bounded above block by block by a convergent series, so its increasing partial sums are bounded and

$$\boxed{\sum_{n=1}^{\infty} e^{-\sqrt{n}} \text{ converges.}}$$