

Question 4

Determine whether $\sum_{n=1}^{\infty} \frac{2n+1}{n^3+1}$ converges or diverges.

- (a) Identify the reciprocal-power benchmark suggested by the leading terms.
- (b) Compute the limit-comparison ratio with that benchmark.
- (c) State the conclusion and give a direct upper comparison valid for $n \geq 1$.

Question 4 – Solution**Step 1: Identify a benchmark.**

The terms are positive. Comparing the highest powers gives $(2n)/(n^3) = 2/n^2$, so the natural benchmark is the convergent p -series $b_n = 1/n^2$.

Step 2: Compute the limit-comparison ratio.

$$\lim_{n \rightarrow \infty} \frac{(2n+1)/(n^3+1)}{1/n^2} = \lim_{n \rightarrow \infty} \frac{2n^3+n^2}{n^3+1} = 2.$$

The ratio limit is 2, which is finite and strictly positive. These are exactly the hypotheses of the Limit Comparison Test. Since $\sum 1/n^2$ converges ($p = 2 > 1$), the given series converges.

Step 3: Verify with a direct inequality.

For $n \geq 1$, $2n+1 \leq 3n$ and $n^3+1 \geq n^3$, so

$$0 < \frac{2n+1}{n^3+1} \leq \frac{3}{n^2}.$$

Because $\sum 3/n^2$ converges, the Direct Comparison Test independently confirms the conclusion. The lower-order terms affect numerical values but not tail behavior.