

Question 3

Determine whether $\sum_{n=1}^{\infty} \frac{\sin^2 n}{n}$ converges or diverges (angles in radians).

- (a) Explain why $0 \leq \sin^2 n/n \leq 1/n$ does not determine convergence.
- (b) Use $\sin^2 n = (1 - \cos 2n)/2$ to split the partial sums.
- (c) Show that $\sum \cos(2n)/n$ converges by the Dirichlet Test, then classify the original series.

Question 3 – Solution

Step 1: Audit the obvious comparison.

The terms satisfy $0 \leq \sin^2 n/n \leq 1/n$ and approach zero. However, an upper comparison with a divergent series is logically inconclusive: a smaller positive series may converge or diverge.

Step 2: Separate the average and oscillatory parts.

Using $\sin^2 n = (1 - \cos 2n)/2$ gives

$$s_N = \frac{1}{2} \sum_{n=1}^N \frac{1}{n} - \frac{1}{2} \sum_{n=1}^N \frac{\cos(2n)}{n}.$$

Step 3: Control the oscillatory series.

To apply Dirichlet's Test to $\sum \cos(2n)/n$, verify both hypotheses. First, the partial sums of $\cos(2n)$ are bounded. Indeed,

$$\sum_{n=1}^N e^{2in} = e^{2i} \frac{1 - e^{2iN}}{1 - e^{2i}},$$

whose magnitude is at most $2/|1 - e^{2i}|$; taking real parts preserves boundedness. Second, $1/n$ is positive, decreasing, and tends to zero. Dirichlet's Test therefore shows that $\sum \cos(2n)/n$ converges.

Step 4: Combine the two parts.

The first part of s_N is one-half of the harmonic partial sum and tends to $+\infty$, while the oscillatory part approaches a finite limit. Subtracting a bounded quantity cannot stop the harmonic growth. Hence

$$\boxed{\sum_{n=1}^{\infty} \frac{\sin^2 n}{n} \text{ diverges.}}$$

The needed extra structure is that $\sin^2 n$ has positive average $1/2$, not merely that it is bounded.