

Question 2

Determine whether $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ converges or diverges.

- (a) Verify that the terms approach zero and explain why that does not settle convergence.
- (b) Group the indices into dyadic blocks $2^k \leq n < 2^{k+1}$.
- (c) Find a lower bound for each block and use it to classify the series.

Question 2 – Solution**Step 1: Check the terms.**

For $n \geq 2$, $a_n = 1/(n \ln n) > 0$, and $n \ln n \rightarrow \infty$. Thus $a_n \rightarrow 0$. The n th-term test is inconclusive, so a tail estimate is required.

Step 2: Form dyadic blocks.

Use the block $2^k \leq n < 2^{k+1}$. It contains exactly 2^k integers. Within this block,

$$n < 2^{k+1}, \quad \ln n < (k+1) \ln 2,$$

and hence

$$\frac{1}{n \ln n} > \frac{1}{2^{k+1}(k+1) \ln 2}.$$

Step 3: Estimate one complete block.

Because the block contains 2^k terms, its sum is greater than

$$2^k \frac{1}{2^{k+1}(k+1) \ln 2} = \frac{1}{2(k+1) \ln 2}.$$

Step 4: Compare the block series.

Summing these lower bounds over k produces the constant multiple $(2 \ln 2)^{-1} \sum 1/(k+1)$ of the divergent harmonic series. Hence the original partial sums are unbounded and

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n} \text{ diverges.}$$

Step 5: Interpret the slow growth.

The hidden continuous substitution is $u = \ln x$, since $du = dx/x$ and $\int dx/(x \ln x) = \ln(\ln x) + C$. This explains the exceptionally slow divergence; a large numerical cutoff can misleadingly suggest stabilization.