

Question 10

For a real parameter r , consider $\sum_{n=1}^{\infty} r^{2n}$.

- (a) Identify the first term and common ratio as functions of r .
- (b) Determine all r for which the series converges and find its sum.
- (c) Analyze the boundary values $r = \pm 1$ and the region $|r| > 1$ using the n th-term test.

Question 10 – Solution**Step 1: Identify the geometric data.**

Expanding the first terms gives

$$r^2 + r^4 + r^6 + \dots,$$

Thus the first term is $a = r^2$ and the common ratio is $q = r^2$. A geometric series converges precisely when $|q| < 1$, so

$$|r^2| < 1 \iff |r| < 1.$$

Step 2: Find the sum on the convergence interval.For $|r| < 1$, the geometric formula gives

$$S(r) = \frac{r^2}{1 - r^2}.$$

The finite partial sum provides a direct verification:

$$s_N = \frac{r^2(1 - r^{2N})}{1 - r^2} \quad (r^2 \neq 1),$$

When $|r| < 1$, $r^{2N} \rightarrow 0$, so $s_N \rightarrow r^2/(1 - r^2)$.**Step 3: Check the endpoints and exterior region.**

At both $r = 1$ and $r = -1$, every term r^{2n} equals 1, so $s_N = N \rightarrow \infty$. If $|r| > 1$, then r^{2n} grows rather than approaches zero; the nth-term test proves divergence. Thus the convergence set is exactly $(-1, 1)$.