

Question 9

Consider $\sum_{n=1}^{\infty} (\sqrt{n+1} - \sqrt{n})$.

- (a) Write the first four terms and expose the telescoping cancellation.
- (b) Derive the N th partial sum and determine whether the series converges.
- (c) Reconcile the divergence with the fact that the individual terms approach zero.

Question 9 – Solution**Step 1: Form the finite partial sum.**

Write the N th partial sum before passing to infinity:

$$\begin{aligned}s_N &= (\sqrt{2} - 1) + (\sqrt{3} - \sqrt{2}) + \cdots + (\sqrt{N+1} - \sqrt{N}) \\ &= \sqrt{N+1} - 1.\end{aligned}$$

Step 2: Analyze the partial-sum limit.

All interior square-root terms cancel, leaving only the first negative boundary term and final positive boundary term. Because $\sqrt{N+1} - 1 \rightarrow \infty$, the partial sums are unbounded and the series diverges to $+\infty$.

Step 3: Check the n th-term condition.

The summand does approach zero. Rationalization gives

$$\sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1} + \sqrt{n}} \rightarrow 0.$$

Step 4: Reconcile the two facts.

There is no contradiction: the n th-term condition is necessary, not sufficient. Here every increment is positive and shrinks to zero, but the accumulated total still grows without bound. Since the denominator above is asymptotic to $2\sqrt{n}$, each increment is asymptotic to $1/(2\sqrt{n})$, consistent with a divergent p -series scale.