

Question 8

Evaluate $\sum_{n=1}^{\infty} \frac{1}{(n+1)!}$.

- (a) Reindex the series using $k = n + 1$.
- (b) Compare it with the Maclaurin series for e and identify every omitted term.
- (c) Find the sum and give a simple upper bound for the remainder after N terms.

Question 8 – Solution**Step 1: Reindex carefully.**

Let $k = n + 1$. When $n = 1$, the new index is $k = 2$, and as $n \rightarrow \infty$, so does k . Therefore

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)!} = \sum_{k=2}^{\infty} \frac{1}{k!}.$$

Step 2: Compare with the exponential series.

The Maclaurin series for $e = e^1$ is

$$e = \sum_{k=0}^{\infty} \frac{1}{k!} = \frac{1}{0!} + \frac{1}{1!} + \sum_{k=2}^{\infty} \frac{1}{k!}.$$

The desired tail begins at $k = 2$, so the omitted terms are exactly $1/0! = 1$ and $1/1! = 1$. Therefore

$$S = e - 2.$$

Step 3: Bound the remainder.

After the original terms through $n = N$, the tail starts at $k = N + 2$. For $j \geq 0$, each new factorial factor is at least $N + 3$, so

$$(N + 2 + j)! \geq (N + 2)!(N + 3)^j.$$

Taking reciprocals and summing gives

$$0 < R_N \leq \frac{1}{(N + 2)!} \sum_{j=0}^{\infty} \frac{1}{(N + 3)^j} = \frac{N + 3}{(N + 2)(N + 2)!}.$$

This bound is explicit, positive, and tends to zero, confirming the convergence of the partial sums to $e - 2$.