

Question 7

Evaluate $\sum_{n=1}^{\infty} \frac{n}{2^n}$.

- (a) Differentiate the geometric-series identity to derive a formula for $\sum_{n=1}^{\infty} nx^{n-1}$ when $|x| < 1$.
- (b) Use that formula at $x = 1/2$ to evaluate the series.
- (c) Derive the finite partial sum and exact remainder after N terms.

Question 7 – Solution

Step 1: Start from the geometric series.

For $|x| < 1$, the geometric identity is

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

Step 2: Differentiate within the interval of convergence.

Power series may be differentiated term by term at every interior point of their interval of convergence. Thus

$$\sum_{n=1}^{\infty} nx^{n-1} = \frac{1}{(1-x)^2}.$$

Multiplying both sides by x aligns the power with the desired form:

$$\sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}.$$

Step 3: Substitute the required value.

Since $|1/2| < 1$, substitution is valid and gives $(1/2)/(1/2)^2 = 2$.

Step 4: Derive the finite sum and remainder.

Differentiating the finite identity $1 + x + \cdots + x^N = (1 - x^{N+1})/(1 - x)$ and multiplying by x yields

$$\sum_{n=1}^N nx^n = \frac{x - (N+1)x^{N+1} + Nx^{N+2}}{(1-x)^2}.$$

At $x = 1/2$, this simplifies to

$$s_N = 2 - \frac{N+2}{2^N}, \quad R_N = 2 - s_N = \frac{N+2}{2^N}.$$