

**Question 5**

Consider  $\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$ .

- (a) Find a partial-fraction decomposition that produces telescoping cancellation.
- (b) Derive the exact  $N$ th partial sum and verify it for  $N = 1, 2, 3$ .
- (c) Find the sum and the exact remainder after  $N$  terms.

**Question 5 – Solution****Step 1: Find the partial fractions.**

Write  $1/[n(n+2)] = A/n + B/(n+2)$ . Multiplying by  $n(n+2)$  gives  $1 = A(n+2) + Bn$ . Matching coefficients yields  $A = 1/2$  and  $B = -1/2$ , so

$$\frac{1}{n(n+2)} = \frac{1}{2} \left( \frac{1}{n} - \frac{1}{n+2} \right).$$

**Step 2: Telescope the finite partial sum.**

Hence

$$s_N = \frac{1}{2} \left( 1 + \frac{1}{2} - \frac{1}{N+1} - \frac{1}{N+2} \right) = \frac{3}{4} - \frac{1}{2(N+1)} - \frac{1}{2(N+2)}.$$

The surviving terms are 1,  $1/2$ ,  $-1/(N+1)$ , and  $-1/(N+2)$ . For  $N = 1, 2, 3$ , the formula gives  $1/3$ ,  $11/24$ , and  $21/40$ , matching direct addition.

**Step 3: Take the limit.**

Since both remaining reciprocal terms approach zero,

$$S = \frac{3}{4}.$$

**Step 4: State the exact error.**

Subtracting the partial sum from  $S$  gives the exact positive tail

$$R_N = S - s_N = \frac{1}{2(N+1)} + \frac{1}{2(N+2)}.$$