

Question 4

Consider $\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right)$.

- (a) Rewrite the N th partial sum as the logarithm of a product.
- (b) Simplify that product and determine whether the series converges.
- (c) Explain how the term test and the growth of the partial sums illustrate different facts.

Question 4 – Solution**Step 1: Work with a finite partial sum.**

For positive factors, $\ln u + \ln v = \ln(uv)$. Therefore

$$s_N = \ln \prod_{n=1}^N \frac{n+1}{n} = \ln \left(\frac{2}{1} \frac{3}{2} \frac{4}{3} \cdots \frac{N+1}{N} \right) = \ln(N+1).$$

Step 2: Simplify the product.

Every interior factor cancels: the numerator $2 \cdot 3 \cdots N$ cancels the same factors in the denominator, leaving $N+1$. This is multiplicative telescoping.

Step 3: Classify the series.

Since $s_N = \ln(N+1) \rightarrow \infty$, the sequence of partial sums is unbounded and the series diverges to $+\infty$.

Step 4: Interpret the nth-term condition.

The individual terms do satisfy

$$\ln \left(\frac{n+1}{n} \right) = \ln(1 + 1/n) \rightarrow 0,$$

which is necessary but not sufficient for convergence. The logarithm grows slowly, explaining why a short numerical table can look nearly stable even though the partial sums are unbounded.