

Question 3

For a real parameter x , consider $\sum_{n=0}^{\infty} x^n$.

- (a) For $x \neq 1$, derive the finite identity $s_N = (1 - x^{N+1})/(1 - x)$.
- (b) Determine every x for which the infinite series converges and find its sum there.
- (c) Analyze $x = 1$, $x = -1$, and $|x| > 1$ separately.

Question 3 – Solution

Step 1: Derive the finite identity.

Let $s_N = 1 + x + \cdots + x^N$. Then $xs_N = x + x^2 + \cdots + x^{N+1}$. Subtracting the second equation from the first gives

$$(1 - x)s_N = 1 - x^{N+1},$$

so for $x \neq 1$ division by $1 - x$ yields

$$s_N = \frac{1 - x^{N+1}}{1 - x}.$$

Step 2: Determine the convergent parameter region.

If $|x| < 1$, then $x^{N+1} \rightarrow 0$. Therefore the partial sums converge and

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1 - x}.$$

Step 3: Check the excluded cases.

At $x = 1$, $s_N = N + 1 \rightarrow \infty$. At $x = -1$, the partial sums alternate between 1 and 0 and have no limit. If $|x| > 1$, the terms x^n do not approach zero, so the nth-term test proves divergence. Thus convergence occurs exactly for $|x| < 1$; checking both endpoints separately is essential.