

Question 2

Consider $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right)$.

- (a) Write out the first four terms and display the cancellation.
- (b) Find a formula for the N th partial sum, identifying the surviving boundary terms.
- (c) Decide whether the series converges and, if so, find its sum and exact remainder.

Question 2 – Solution**Step 1: Form a finite partial sum.**

Never telescope directly to infinity. First write the N th partial sum:

$$\begin{aligned}s_N &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{N} - \frac{1}{N+1}\right) \\ &= 1 - \frac{1}{N+1}.\end{aligned}$$

Step 2: Identify the surviving boundary terms.

Every interior fraction appears once positively and once negatively, so it cancels. Only the initial 1 and final $-1/(N+1)$ survive.

Step 3: Take the partial-sum limit.

By definition, the series converges exactly when (s_N) has a finite limit. Here

$$S = \lim_{N \rightarrow \infty} s_N = 1.$$

Step 4: Compute and verify the remainder.

The exact tail after N terms is

$$R_N = S - s_N = \frac{1}{N+1}.$$

As an algebraic check, $s_N - s_{N-1} = 1/N - 1/(N+1)$, which reproduces the original N th summand exactly.