

Question 1

Consider the series $\sum_{n=1}^{\infty} \frac{3}{4^n}$.

- (a) Identify the first term and common ratio, then evaluate the series using the infinite geometric-series formula.
- (b) Derive a formula for the N th partial sum s_N and evaluate $\lim_{N \rightarrow \infty} s_N$.
- (c) Find the exact remainder $R_N = S - s_N$ and determine the smallest N for which $R_N < 10^{-4}$.

Question 1 – Solution**Step 1: Identify the geometric structure.**

Writing the first terms as $3/4 + 3/16 + 3/64 + \cdots$ shows that the first term is $a = 3/4$ and the common ratio is $r = 1/4$. Since $|r| < 1$, the infinite geometric-series formula applies:

$$S = \frac{a}{1-r} = \frac{3/4}{1-1/4} = 1.$$

Step 2: Confirm the result from partial sums.

For a geometric series, the finite sum is $s_N = a(1 - r^N)/(1 - r)$. Substitution gives

$$s_N = \frac{3}{4} \frac{1 - (1/4)^N}{1 - 1/4} = 1 - 4^{-N},$$

Since $4^{-N} \rightarrow 0$, $s_N \rightarrow 1$, confirming the result from the definition of an infinite series.

Step 3: Find and use the exact remainder.

The exact remainder after N terms is

$$R_N = S - s_N = 4^{-N}.$$

We require $4^{-N} < 10^{-4}$, equivalently $4^N > 10^4$. Since $4^6 = 4096 \leq 10^4$ but $4^7 = 16384 > 10^4$, the smallest choice is $N = 7$. The strict inequality is important.