

Question 10

Define $a_1 = 0$ and $a_{n+1} = \frac{1}{2 + a_n}$.

- (a) Show that $0 \leq a_n \leq 1/2$ for all n .
- (b) Find the unique fixed point L in this interval and prove that consecutive terms alternate around it.
- (c) Establish an error contraction and use it to prove $a_n \rightarrow L$.

Question 10 – Solution**Step 1: Establish an invariant interval.**

The bound follows by induction. The base case $a_1 = 0$ lies in $[0, 1/2]$. If $0 \leq a_n \leq 1/2$, then $2 + a_n \geq 2$, so $0 < a_{n+1} = 1/(2 + a_n) \leq 1/2$. Hence every term remains in $[0, 1/2]$.

Step 2: Find the admissible fixed point.

A fixed point satisfies

$$L = \frac{1}{2 + L} \iff L^2 + 2L - 1 = 0,$$

so $L = -1 \pm \sqrt{2}$. Only $L = \sqrt{2} - 1$ lies in the invariant interval $[0, 1/2]$.

Step 3: Explain the alternating behavior.

Let $f(x) = 1/(2 + x)$. Because $f'(x) = -1/(2 + x)^2 < 0$, f is strictly decreasing. Since $a_1 < L$, $a_2 = f(a_1) > f(L) = L$. Applying the decreasing map reverses the side of L at every iteration, so successive terms alternate across the fixed point.

Step 4: Prove convergence with an error contraction.

$$|a_{n+1} - L| = \left| \frac{1}{2 + a_n} - \frac{1}{2 + L} \right| = \frac{|a_n - L|}{(2 + a_n)(2 + L)} \leq \frac{1}{4} |a_n - L|.$$

Both denominator factors are at least 2, which justifies the factor $1/4$. Iterating gives

$$|a_n - L| \leq \left(\frac{1}{4}\right)^{n-1} |a_1 - L| \longrightarrow 0.$$

Therefore $a_n \rightarrow \sqrt{2} - 1$; the even and odd subsequences approach the same value from opposite sides.