

Question 9

For $n \geq 2$, let $a_n = \left(1 - \frac{1}{n}\right)^{n^2}$.

- (a) Take logarithms and use $\ln(1 - x) \leq -x$ for $0 < x < 1$ to obtain a useful bound.
- (b) Prove that $a_n \rightarrow 0$.
- (c) Use $\ln(1 - x) = -x - x^2/2 + O(x^3)$ to describe the rate of decay more precisely.

Question 9 – Solution

Step 1: Take logarithms.

For $n \geq 2$, $0 < 1 - 1/n < 1$, so $a_n > 0$ and logarithms are valid:

$$\ln a_n = n^2 \ln \left(1 - \frac{1}{n} \right).$$

Step 2: Obtain a decisive upper bound.

For $0 < x < 1$, $\ln(1-x) \leq -x$. With $x = 1/n$, this gives $\ln a_n \leq -n$. Because the exponential function is increasing,

$$0 < a_n \leq e^{-n} \longrightarrow 0.$$

The Squeeze Theorem proves $a_n \rightarrow 0$.

Step 3: Refine the decay rate.

For a sharper description, substitute $x = 1/n$ into $\ln(1-x) = -x - x^2/2 + O(x^3)$:

$$\ln a_n = n^2 \left(-\frac{1}{n} - \frac{1}{2n^2} + O(n^{-3}) \right) = -n - \frac{1}{2} + O(n^{-1}).$$

Exponentiating both sides gives

$$a_n = e^{-n-1/2+O(1/n)} \sim e^{-n-1/2}.$$

The notation $u_n \sim v_n$ means $u_n/v_n \rightarrow 1$. Thus the quadratic exponent repeats a base close to 1 often enough to produce decay essentially like e^{-n} , with the additional constant factor $e^{-1/2}$.