

Question 8

Let $a_n = \int_0^1 x^n dx$ for $n \geq 1$.

- (a) Evaluate the integral and find $\lim_{n \rightarrow \infty} a_n$.
- (b) Prove directly from the integrands that (a_n) is decreasing.
- (c) Give a geometric explanation for why the areas approach zero, despite $x^n = 1$ at $x = 1$.

Question 8 – Solution

Step 1: Evaluate the area exactly.

By the power rule,

$$a_n = \left[\frac{x^{n+1}}{n+1} \right]_0^1 = \frac{1}{n+1} \rightarrow 0.$$

Step 2: Prove monotonicity directly from the integrands.

For $0 \leq x \leq 1$, $0 \leq x^{n+1} \leq x^n$, with strict inequality on $(0, 1)$. Integrating over $[0, 1]$ gives $0 < a_{n+1} < a_n$, so the areas decrease.

Step 3: Explain the geometry rigorously.

For every fixed $x \in [0, 1)$, $x^n \rightarrow 0$; only the single endpoint $x = 1$ remains at height 1, and one point has zero width and contributes no area. For a quantitative version, fix $0 < c < 1$ and split the interval:

$$\int_0^1 x^n dx = \int_0^c x^n dx + \int_c^1 x^n dx \leq c^n + (1 - c).$$

Given $\varepsilon > 0$, first choose c so that $1 - c < \varepsilon/2$. Then choose n so large that $c^n < \varepsilon/2$. The displayed bound gives $a_n < \varepsilon$, proving geometrically that the areas vanish.