

Question 7

Let $a_n = \sqrt{n+1} - \sqrt{n}$.

- (a) Rationalize the expression and use the result to find the limit.
- (b) Prove that (a_n) is positive and decreasing.
- (c) Show that $a_n \sim 1/(2\sqrt{n})$ and interpret this as the spacing between neighboring square roots.

Question 7 – Solution**Step 1: Rationalize the difference.**

Multiplying numerator and denominator by the conjugate avoids the indeterminate form $\infty - \infty$:

$$a_n = \frac{(n+1) - n}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1} + \sqrt{n}}.$$

Step 2: Find the limit and monotonicity.

The denominator tends to infinity, so $a_n \rightarrow 0$. It is also positive. Furthermore, $\sqrt{n+2} + \sqrt{n+1} > \sqrt{n+1} + \sqrt{n}$; taking reciprocals of positive quantities reverses the inequality and gives $a_{n+1} < a_n$. Thus the sequence decreases to zero.

Step 3: Determine the asymptotic gap size.

Compare a_n with $1/(2\sqrt{n})$ by taking their ratio:

$$\frac{a_n}{1/(2\sqrt{n})} = \frac{2\sqrt{n}}{\sqrt{n+1} + \sqrt{n}} = \frac{2}{\sqrt{1+1/n} + 1} \rightarrow 1.$$

Because the ratio tends to 1, $a_n \sim 1/(2\sqrt{n})$. Therefore neighboring square roots get closer, and their gap is approximately $1/(2\sqrt{n})$ for large n .