

Question 6

Let $a_n = \frac{(-1)^n}{n}$.

- (a) Prove that $a_n \rightarrow 0$ using an absolute-value estimate.
- (b) Show that the full sequence is not monotone and is not eventually monotone.
- (c) Describe the monotonic behavior of its even and odd subsequences.

Question 6 – Solution**Step 1: Prove convergence.**

Use absolute values to remove the alternating sign:

$$|a_n - 0| = \frac{1}{n} \longrightarrow 0,$$

Since $1/n \rightarrow 0$, the definition of convergence gives $a_n \rightarrow 0$. Equivalently, $-1/n \leq a_n \leq 1/n$ and the Squeeze Theorem applies.

Step 2: Test eventual monotonicity.

For every positive integer k ,

$$a_{2k} = \frac{1}{2k} > 0, \quad a_{2k+1} = -\frac{1}{2k+1} < 0.$$

Thus $a_{2k} > a_{2k+1}$ but $a_{2k+1} < a_{2k+2}$, producing infinitely many decreases and increases. Because this occurs for arbitrarily large k , no tail can be monotone.

Step 3: Analyze the two subsequences.

The even subsequence $a_{2k} = 1/(2k)$ decreases to 0, while the odd subsequence $a_{2k-1} = -1/(2k-1)$ increases to 0. Both close in on the same limit from opposite sides. This demonstrates that convergence does not require eventual monotonicity.