

Question 5

Let $a_n = \frac{n^2}{2^n}$ for $n \geq 1$.

- (a) Use a_{n+1}/a_n to determine exactly when $a_{n+1} < a_n$.
- (b) Identify the largest term and the first index from which the sequence is strictly decreasing.
- (c) Prove that $a_n \rightarrow 0$.

Question 5 – Solution**Step 1: Compare consecutive positive terms.**

Because $a_n > 0$, the inequality $a_{n+1} < a_n$ is equivalent to $a_{n+1}/a_n < 1$. Compute

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^2}{2n^2}.$$

Step 2: Solve the ratio inequality exactly.

Thus $a_{n+1} < a_n$ exactly when

$$(n+1)^2 < 2n^2 \iff n^2 - 2n - 1 > 0 \iff n > 1 + \sqrt{2}.$$

For integer $n \geq 1$, this is equivalent to $n \geq 3$. Since $a_1 = 1/2 < a_2 = 1 < a_3 = 9/8$ and $a_{n+1} < a_n$ for every $n \geq 3$, a_3 is the unique largest term and the sequence is strictly decreasing from index 3 onward.

Step 3: Prove the decreasing tail approaches zero.

A decreasing positive sequence has a limit, but the ratio bound identifies it directly. For $n \geq 7$,

$$\frac{a_{n+1}}{a_n} = \frac{1}{2} \left(1 + \frac{1}{n}\right)^2 \leq \frac{1}{2} \left(\frac{8}{7}\right)^2 = \frac{32}{49} < 1.$$

Iterating this inequality gives $0 < a_n \leq a_7(32/49)^{n-7}$ for $n \geq 7$. The right side is geometric with ratio less than 1, so it tends to zero. The Squeeze Theorem yields $a_n \rightarrow 0$.