

Question 4

Let $a_n = \sum_{k=1}^n 2^{-k}$.

- (a) Derive a closed formula for a_n from the finite geometric-sum formula.
- (b) Prove that (a_n) converges and find its limit.
- (c) Find the exact remaining gap $L - a_n$ and the smallest n for which this gap is below 10^{-3} .

Question 4 – Solution

Step 1: Identify and sum the finite geometric series.

The terms form a geometric sequence with first term $a = 1/2$ and common ratio $r = 1/2$. Therefore

$$a_n = \frac{(1/2)(1 - (1/2)^n)}{1 - 1/2} = 1 - 2^{-n}.$$

Step 2: Determine convergence and the limit.

Since $2^{-n} \rightarrow 0$, the closed formula gives $a_n \rightarrow L = 1$. Also $a_{n+1} - a_n = 2^{-(n+1)} > 0$ and $a_n < 1$, so the sequence is increasing and bounded above, independently confirming convergence by the Monotone Convergence Theorem.

Step 3: Compute the exact remaining gap.

The exact error after n terms is

$$L - a_n = 2^{-n}.$$

Step 4: Solve the accuracy inequality.

We need $2^{-n} < 10^{-3}$, equivalently $2^n > 1000$. Since $2^9 = 512 \leq 1000$ but $2^{10} = 1024 > 1000$, the smallest qualifying index is $n = 10$.