

Question 3

Let $a_n = \{n\sqrt{2}\} = n\sqrt{2} - \lfloor n\sqrt{2} \rfloor$.

- (a) Prove that $0 \leq a_n < 1$, so the sequence is bounded.
- (b) Using the fact that the fractional parts of multiples of an irrational number are dense in $[0, 1]$, find two different subsequential limits.
- (c) Decide whether (a_n) converges and explain why boundedness is insufficient.

Question 3 – Solution**Step 1: Establish boundedness.**

For every real x , its fractional part satisfies $0 \leq \{x\} < 1$. Therefore $0 \leq a_n < 1$ for all n , and the sequence is bounded.

Step 2: Use irrationality and density.

Because $\sqrt{2}$ is irrational, the density theorem for irrational rotations says that the set $\{n\sqrt{2}\} : n \geq 1\}$ is dense in $[0, 1]$. In particular, every open subinterval contains infinitely many terms, allowing the selection of increasing subsequence indices.

Step 3: Construct two subsequences.

For each positive integer j , choose increasing indices n_j and m_j such that

$$0 < a_{n_j} < 1/j, \quad 1/2 - 1/j < a_{m_j} < 1/2 + 1/j.$$

The first inequality and Squeeze Theorem give $a_{n_j} \rightarrow 0$; the second gives $a_{m_j} \rightarrow 1/2$.

Step 4: Apply the subsequence criterion.

Every subsequence of a convergent sequence must approach the same limit as the original sequence. Since these two subsequences have different limits, (a_n) diverges. Boundedness alone guarantees neither monotonicity nor convergence.