

Question 2

Let (b_n) be a bounded real sequence and define $a_n = \sup\{b_k : k \geq n\}$.

- (a) Explain why every a_n exists.
- (b) Prove that (a_n) is decreasing and bounded, and hence convergent.
- (c) State what its limit represents, and compute it when $b_n = (-1)^n + 1/n$.

Question 2 – Solution

Step 1: Verify that each supremum exists.

Let $T_n = \{b_k : k \geq n\}$. Each tail is nonempty, and because (b_n) is bounded, T_n is bounded above. The least-upper-bound property of $\mathbb{R}$ therefore guarantees that $a_n = \sup T_n$ exists.

Step 2: Prove monotonicity.

Removing the first element of a tail cannot increase its set of possible values. Formally, $T_{n+1} \subseteq T_n$, so

$$a_{n+1} = \sup T_{n+1} \leq \sup T_n = a_n.$$

Thus (a_n) is decreasing.

Step 3: Establish a lower bound and convergence.

If m is any lower bound for (b_n) , then $m \leq b_k \leq a_n$ for every $k \geq n$, so $m \leq a_n$. Hence (a_n) is bounded below. The Monotone Convergence Theorem now gives convergence, and its limit is called the limit superior:

$$\lim_{n \rightarrow \infty} a_n = \limsup_{n \rightarrow \infty} b_n.$$

Step 4: Apply the construction to the example.

For $b_n = (-1)^n + 1/n$, even terms equal $1 + 1/(2j)$ and odd terms are negative. Among the even terms in a tail, the earliest one is largest. If e_n is the first even integer satisfying $e_n \geq n$, then

$$a_n = 1 + \frac{1}{e_n} \longrightarrow 1.$$

Thus $\limsup b_n = 1$. The tail-supremum process has turned oscillating bounded data into a decreasing upper envelope.