

**Question 1**

Define  $a_1 = 2$  and  $a_{n+1} = \frac{1}{2} \left( a_n + \frac{3}{a_n} \right)$ .

- (a) Explain why this is Newton's method applied to  $x^2 - 3 = 0$ .
- (b) Prove that  $a_n \geq \sqrt{3}$  for every  $n$ , and prove that  $(a_n)$  is decreasing.
- (c) Conclude that the sequence converges and determine its limit.

**Question 1 – Solution****Step 1: Identify the algorithm.**

For  $f(x) = x^2 - 3$ , Newton's method uses  $x_{n+1} = x_n - f(x_n)/f'(x_n)$ . Since  $f'(x) = 2x$ , the update becomes

$$x - \frac{f(x)}{f'(x)} = x - \frac{x^2 - 3}{2x} = \frac{1}{2} \left( x + \frac{3}{x} \right),$$

which is exactly the given recurrence.

**Step 2: Establish positivity and a lower bound.**

All terms are positive because  $a_1 = 2 > 0$  and, whenever  $a_n > 0$ , both  $a_n$  and  $3/a_n$  are positive. More precisely,

$$a_{n+1} - \sqrt{3} = \frac{(a_n - \sqrt{3})^2}{2a_n} \geq 0,$$

The right side is nonnegative, so  $a_{n+1} \geq \sqrt{3}$ . Since  $a_1 \geq \sqrt{3}$ , induction proves  $a_n \geq \sqrt{3}$  for every  $n$ .

**Step 3: Prove monotonicity.**

Using the lower bound,

$$a_{n+1} - a_n = \frac{3 - a_n^2}{2a_n} \leq 0,$$

Thus  $a_{n+1} \leq a_n$ , so  $(a_n)$  is decreasing.

**Step 4: Prove convergence before solving for the limit.**

The sequence is decreasing and bounded below by  $\sqrt{3}$ . The Monotone Convergence Theorem therefore guarantees  $a_n \rightarrow L$  for some  $L \geq \sqrt{3}$ . Continuity and  $L > 0$  allow passage to the limit:

$$L = \frac{1}{2} \left( L + \frac{3}{L} \right), \quad L^2 = 3.$$

The equation gives  $L = \pm\sqrt{3}$ , and the established positive lower bound selects  $L = \sqrt{3}$ .