

Question 10

Find the binomial series, radius, and exact interval of convergence for $(1 + 3x)^{-1/2}$. Test $x = -1/3$ and $x = 1/3$ separately.

Question 10 – Solution

Step 1: Substitute into the parent series.

$$(1 + u)^{-1/2} = \sum_{n=0}^{\infty} \binom{-1/2}{n} u^n.$$

With $u = 3x$,

$$(1 + 3x)^{-1/2} = \sum_{n=0}^{\infty} \binom{-1/2}{n} 3^n x^n = 1 - \frac{3}{2}x + \frac{27}{8}x^2 - \frac{135}{16}x^3 + \cdots.$$

Step 2: Find the radius.

The condition $|u| < 1$ becomes $|3x| < 1$, so $R = 1/3$.

Step 3: Test endpoints.

At $x = 1/3$, $u = 1$, and the terms alternate with magnitude asymptotic to $1/\sqrt{\pi n}$; the series converges conditionally. At $x = -1/3$, $u = -1$ cancels the coefficient signs, leaving positive terms comparable with $1/\sqrt{n}$; it diverges.

The endpoint magnitudes $b_n = 4^{-n} \binom{2n}{n}$ decrease, since $b_{n+1}/b_n = (2n+1)/(2n+2) < 1$, and tend to zero. This verifies the alternating-test hypotheses.

Conclusion.

The interval is $\boxed{(-1/3, 1/3]}$.