

Question 6

Derive the binomial series for $(1 - x)^{-2}$ and show that it agrees with differentiating the geometric series. Determine the exact interval of convergence.

Question 6 – Solution**Step 1: Use generalized coefficients.**For $n \geq 0$,

$$\binom{-2}{n} = \frac{(-2)(-3)\cdots(-n-1)}{n!} = (-1)^n(n+1).$$

Substitute $u = -x$ into $(1+u)^{-2}$:

$$\binom{-2}{n}(-x)^n = (-1)^n(n+1)(-1)^n x^n = (n+1)x^n.$$

Thus

$$(1-x)^{-2} = \sum_{n=0}^{\infty} (n+1)x^n.$$

Step 2: Verify by differentiation.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \Rightarrow \quad \frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1} = \sum_{n=0}^{\infty} (n+1)x^n.$$

Step 3: Check convergence.

The radius is 1. At $x = 1$ terms are $n+1$; at $x = -1$ their magnitudes are $n+1$. Both endpoints diverge.

Conclusion.The interval is $\boxed{(-1, 1)}$.