

Question 3

Find the first four nonzero terms of $(1 + x)^{-1/2}$ using generalized binomial coefficients. Explain the signs and determine the exact interval of convergence.

Question 3 – Solution**Step 1: Compute coefficients.**

$$\binom{-1/2}{1} = -\frac{1}{2}, \quad \binom{-1/2}{2} = \frac{3}{8}, \quad \binom{-1/2}{3} = -\frac{5}{16}.$$

Every factor in the numerator is negative, so the coefficient sign is $(-1)^n$. Hence

$$(1+x)^{-1/2} = 1 - \frac{x}{2} + \frac{3x^2}{8} - \frac{5x^3}{16} + \frac{35x^4}{128} - \cdots.$$

Step 2: Find the radius.

The generalized binomial series has $R = 1$.

Step 3: Test endpoints.

The coefficient magnitudes satisfy

$$\left| \binom{-1/2}{n} \right| = \frac{1}{4^n} \binom{2n}{n} \sim \frac{1}{\sqrt{\pi n}}.$$

At $x = 1$ the signs alternate, so the series converges conditionally. At $x = -1$, all terms are positive and comparable with $1/\sqrt{n}$, so it diverges.

The endpoint magnitudes $b_n = 4^{-n} \binom{2n}{n}$ decrease, since $b_{n+1}/b_n = (2n+1)/(2n+2) < 1$, and tend to zero. This verifies the alternating-test hypotheses.

Conclusion.

The interval is $\boxed{(-1, 1]}$.