

Question 9

A principal P earns annual rate r compounded n times per year, so $A_n = P(1 + r/n)^n$. Use the series for $\log(1 + u)$ to derive the continuous-compounding limit as $n \rightarrow \infty$.

Question 9 – Solution**Step 1: Take a logarithm.**

For sufficiently large n , $1 + r/n > 0$, and

$$\log\left(\frac{A_n}{P}\right) = n \log\left(1 + \frac{r}{n}\right).$$

Step 2: Insert the logarithmic expansion.

$$\begin{aligned} n \log\left(1 + \frac{r}{n}\right) &= n \left(\frac{r}{n} - \frac{r^2}{2n^2} + \frac{r^3}{3n^3} + O(n^{-4}) \right) \\ &= r - \frac{r^2}{2n} + \frac{r^3}{3n^2} + O(n^{-3}) \longrightarrow r. \end{aligned}$$

Step 3: Exponentiate.

Continuity of the exponential function gives

$$\frac{A_n}{P} \longrightarrow e^r, \quad \boxed{A_n \longrightarrow P e^r}.$$

The correction $-r^2/(2n)$ also explains the approach rate. For positive r , discrete compounding approaches the continuous value from below.