

Question 8

The small-angle model replaces $\sin \theta$ by θ when θ is measured in radians. At $\theta = 0.2$, compute the relative error $|\theta - \sin \theta|/|\sin \theta|$ and explain it using the sine series.

Question 8 – Solution**Step 1: Use the local series.**

$$\sin \theta = \theta - \frac{\theta^3}{6} + \frac{\theta^5}{120} - \cdots.$$

Thus the leading absolute error in replacing $\sin \theta$ by θ is approximately $\theta^3/6$, and the leading relative error is approximately $\theta^2/6$.

Step 2: Evaluate the exact relative error.

At $\theta = 0.2$ radians,

$$\sin(0.2) \approx 0.1986693308, \quad 0.2 - \sin(0.2) \approx 0.0013306692.$$

Therefore

$$\frac{0.2 - \sin(0.2)}{\sin(0.2)} \approx 0.0066979 = \boxed{0.6698\%}.$$

Step 3: Compare with the leading prediction.

$$\frac{\theta^2}{6} = \frac{0.04}{6} \approx 0.0066667 = 0.6667\%,$$

which is close to the exact relative error. This quantifies why the approximation is useful but not exact.