

Question 7

Derive one forward Euler step for $y' = f(t, y)$ from Taylor's theorem. Identify the discarded term. Then apply one step with $h = 0.1$ to $y' = y$, $y(0) = 1$, and bound its local error.

Question 7 – Solution**Step 1: Expand the exact solution in time.**Taylor's theorem about t_n gives

$$y(t_n + h) = y(t_n) + hy'(t_n) + \frac{h^2}{2}y''(\xi_n)$$

for some ξ_n between t_n and $t_n + h$.**Step 2: Replace the derivative using the ODE.**Since $y'(t_n) = f(t_n, y(t_n))$, dropping the quadratic remainder produces Euler's update

$$\boxed{y_{n+1} = y_n + hf(t_n, y_n)}.$$

The one-step local truncation error is $h^2y''(\xi_n)/2$, hence is order h^2 .**Step 3: Apply the method.**For $y' = y$, $y_0 = 1$, and $h = 0.1$,

$$y_1 = 1 + 0.1(1) = \boxed{1.1}.$$

The exact value is $e^{0.1}$. On $[0, 0.1]$, $y''(t) = e^t \leq e^{0.1}$, so

$$|e^{0.1} - 1.1| \leq \frac{e^{0.1}(0.1)^2}{2} < 0.00553.$$