

Question 6

For $0 < x \leq 1$, compare the linear approximation $L(x) = x$ with $\log(1 + x)$. Decide whether L overestimates or underestimates, and use the alternating series to bound the absolute error.

Question 6 – Solution

Step 1: Expand the logarithm.

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots, \quad 0 < x \leq 1.$$

The first neglected term after the linear term is negative.

Step 2: Determine the direction.

Subtract the function from the linear approximation:

$$x - \log(1+x) = \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4} - \cdots.$$

This is positive by the Alternating Series Test. Therefore x is an **overestimate** of $\log(1+x)$.

Step 3: Bound the error.

The alternating remainder after the first term is no larger than the first omitted magnitude:

$$0 < x - \log(1+x) \leq \frac{x^2}{2}.$$

For example, at $x = 0.2$, the error is at most 0.02; the actual error is approximately 0.01768.