

Question 5

Estimate $I = \int_0^{0.5} e^{-x^2} dx$ by integrating the first four terms of the Maclaurin series for e^{-x^2} . Give a rigorous error bound from the first omitted integrated term.

Question 5 – Solution

Step 1: Substitute into the exponential series.

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \cdots.$$

For $0 \leq x \leq 0.5$, the alternating term magnitudes decrease.

Step 2: Integrate through the x^6 term.

$$\begin{aligned} I &\approx \left[x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} \right]_0^{0.5} \\ &= 0.5 - \frac{0.5^3}{3} + \frac{0.5^5}{10} - \frac{0.5^7}{42} \\ &= \boxed{0.4612723214}. \end{aligned}$$

Step 3: Bound the integrated remainder.

The omitted integrand has magnitude at most $x^8/4!$, and the integrated alternating remainder satisfies

$$|R| \leq \int_0^{0.5} \frac{x^8}{4!} dx = \frac{0.5^9}{9 \cdot 4!} < 9.05 \times 10^{-6}.$$

Thus the estimate is accurate to within 0.00000905.