

Question 4

Solve the initial-value problem $y' = y$, $y(0) = 1$ by assuming a power-series solution $y = \sum_{n=0}^{\infty} a_n x^n$. Derive the coefficient recursion and identify the resulting function.

Question 4 – Solution

Step 1: Differentiate and align powers.

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n.$$

The original series is $y = \sum_{n=0}^{\infty} a_n x^n$.

Step 2: Match coefficients in $y' = y$.

$$(n+1)a_{n+1} = a_n, \quad a_{n+1} = \frac{a_n}{n+1}.$$

The initial condition gives $a_0 = y(0) = 1$. Repeated use of the recursion yields

$$a_1 = 1, \quad a_2 = \frac{1}{2!}, \quad a_3 = \frac{1}{3!}, \quad \text{and in general} \quad a_n = \frac{1}{n!}.$$

Conclusion.

$$y(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = \boxed{e^x}.$$

The series has infinite radius by the Ratio Test, so the solution is valid for every real x .