

Question 3

Derive the Maclaurin series for $\cos x$ from its derivative cycle. Explain why every odd coefficient vanishes, give the degree-6 polynomial, and state the convergence domain.

Question 3 – Solution

Step 1: Evaluate the derivative cycle at zero.

$$\cos x, \quad -\sin x, \quad -\cos x, \quad \sin x, \dots$$

gives values $1, 0, -1, 0, 1, \dots$ at 0. Therefore every odd-order coefficient is zero, and the even coefficients alternate in sign.

Step 2: Write the series and polynomial.

$$\boxed{\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}}, \quad P_6(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}.$$

Step 3: Justify convergence to cosine.

Consecutive absolute terms have ratio

$$\frac{x^2}{(2n+1)(2n+2)} \rightarrow 0,$$

so the series converges for all x . Since every derivative of cosine has magnitude at most 1, the Lagrange remainder is at most $|x|^{m+1}/(m+1)! \rightarrow 0$. Hence the series equals $\cos x$ on $(-\infty, \infty)$.