

Question 2

Derive the Maclaurin series for $\sin x$ from its derivative cycle. Explain why only odd powers occur, give the degree-7 polynomial, and state the convergence domain.

Question 2 – Solution

Step 1: List the derivative cycle.

$$\sin x, \quad \cos x, \quad -\sin x, \quad -\cos x, \quad \sin x, \dots$$

At $x = 0$, the values are $0, 1, 0, -1, 0, \dots$. All even-order derivatives vanish, while the odd derivatives alternate between 1 and -1 .

Step 2: Form the series.

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}.$$

The requested polynomial is

$$P_7(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}.$$

Step 3: Establish convergence.

The absolute-value ratio of consecutive displayed terms is

$$\frac{x^2}{(2n+2)(2n+3)} \longrightarrow 0$$

for every fixed x . Taylor's remainder is bounded by $|x|^{m+1}/(m+1)!$ because all derivatives of sine have magnitude at most 1, so it tends to 0. Thus the series equals $\sin x$ for every real x .