

**Question 1**

Construct the Maclaurin series for  $e^x$  from its derivatives. Prove that it converges to  $e^x$  for every real  $x$ , and give a uniform Lagrange-remainder bound for the degree- $N$  polynomial on  $|x| \leq 1$ .

**Question 1 – Solution**

**Step 1: Compute the coefficients.**

Every derivative of  $f(x) = e^x$  is  $e^x$ , so  $f^{(n)}(0) = 1$ . Hence

$$P_N(x) = \sum_{n=0}^N \frac{x^n}{n!}, \quad \boxed{e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}}.$$

**Step 2: Bound the remainder.**

Taylor's theorem gives

$$R_N(x) = \frac{e^c x^{N+1}}{(N+1)!}$$

for some  $c$  between 0 and  $x$ . If  $|x| \leq 1$ , then  $c \leq 1$ , so

$$\boxed{|R_N(x)| \leq \frac{e|x|^{N+1}}{(N+1)!} \leq \frac{e}{(N+1)!}}.$$

**Step 3: Justify equality with the function.**

For any fixed real  $x$ ,  $e^c \leq e^{|x|}$  and  $e^{|x|}|x|^{N+1}/(N+1)! \rightarrow 0$  by the Ratio Test. Thus  $R_N(x) \rightarrow 0$  for every  $x$ , proving the series represents  $e^x$  on all of  $\mathbb{R}$ .