

Question 8

Find a power series for $F(x) = \int_0^x \frac{dt}{1-t^4}$. Justify term-by-term integration, then determine the radius and exact interval of convergence of the resulting series.

Question 8 – Solution**Step 1: Expand the integrand.**For $|t| < 1$,

$$\frac{1}{1-t^4} = \sum_{n=0}^{\infty} t^{4n}.$$

On every closed subinterval inside $(-1, 1)$, this power series converges uniformly, so it may be integrated term by term.

Step 2: Integrate.

$$F(x) = \sum_{n=0}^{\infty} \int_0^x t^{4n} dt = \boxed{\sum_{n=0}^{\infty} \frac{x^{4n+1}}{4n+1}}.$$

Step 3: Test the endpoints.

Integration preserves radius $R = 1$. At $x = 1$, the series is $\sum 1/(4n+1)$, which diverges by Limit Comparison with $1/n$. At $x = -1$, it is the negative of the same divergent positive series.

Conclusion.The interval is $\boxed{(-1, 1)}$.