

Question 10

For $S = \sum_{n=1}^{\infty} 2^{-n^2}$ and $S_N = \sum_{n=1}^N 2^{-n^2}$, bound $R_N = S - S_N$ by a geometric series. Clearly identify its first term and common ratio.

Question 10 – Solution**Step 1: Reindex the tail.**

Write $n = N + 1 + k$, where $k \geq 0$. Then

$$(N + 1 + k)^2 = (N + 1)^2 + k(2N + 2) + k^2 \geq (N + 1)^2 + k(2N + 3),$$

because $k^2 \geq k$ for integers $k \geq 0$.

Step 2: Compare term by term.

$$2^{-(N+1+k)^2} \leq 2^{-(N+1)^2} \left(2^{-(2N+3)}\right)^k.$$

The comparison series has first term $2^{-(N+1)^2}$ and ratio $q = 2^{-(2N+3)}$.

Step 3: Sum the geometric majorant.

$$0 < R_N \leq \frac{2^{-(N+1)^2}}{1 - 2^{-(2N+3)}}.$$

The bound is very close to the first omitted term because q becomes extremely small.