

Question 7

For $S = \sum_{n=2}^{\infty} \frac{1}{n(\log n)^2}$, derive a two-sided Integral Test estimate for $R_N = S - S_N$. Then determine what the upper bound requires to guarantee $R_N < 0.01$ and interpret the result.

Question 7 – Solution**Step 1: Verify the setup.**

The function $f(x) = 1/[x(\log x)^2]$ is positive and decreasing for $x \geq 2$.

Step 2: Integrate the tail.

With $u = \log x$,

$$\int_A^\infty \frac{dx}{x(\log x)^2} = \int_{\log A}^\infty u^{-2} du = \frac{1}{\log A}.$$

Therefore

$$\boxed{\frac{1}{\log(N+1)} \leq R_N \leq \frac{1}{\log N}}.$$

Step 3: Solve the requested tolerance.

$$\frac{1}{\log N} < 0.01 \iff \log N > 100 \iff N > e^{100}.$$

Thus the least certified integer is $\lfloor e^{100} \rfloor + 1$, an astronomically large index (about 2.69×10^{43}).

Interpretation.

The lower and upper bounds are asymptotically equal, so this is not merely a loose estimate: the series itself converges extraordinarily slowly.