

Question 5

For $S = \sum_{n=0}^{\infty} (1/3)^n$, define $S_N = \sum_{n=0}^N (1/3)^n$.

- (a) Compute the tail $R_N = S - S_N$ exactly.
- (b) Compare it with the standard geometric-tail estimate based on the first omitted term.

Question 5 – Solution

Step 1: Identify the first omitted term.

Because S_N ends at index N , the tail begins at $n = N + 1$:

$$R_N = \sum_{n=N+1}^{\infty} \left(\frac{1}{3}\right)^n.$$

Step 2: Sum the geometric tail.

$$R_N = \frac{(1/3)^{N+1}}{1 - 1/3} = \frac{1}{2 \cdot 3^N}.$$

Thus the exact error is $\boxed{R_N = 3^{-N}/2}$.

Step 3: Compare with the general bound.

For a geometric tail with ratio magnitude $r < 1$,

$$|R_N| \leq \frac{|a_{N+1}|}{1 - r}.$$

Here $a_{N+1} = 3^{-(N+1)}$ and $r = 1/3$, producing $1/(2 \cdot 3^N)$. Because every term is positive and the ratio is exactly constant, the usual “bound” equals the exact tail.