

Question 2

For $S = \sum_{n=1}^{\infty} 1/n^2$, let $S_N = \sum_{n=1}^N 1/n^2$ and $R_N = S - S_N$. Use both sides of the Integral Test remainder estimate to bracket R_N , then bracket S in terms of S_N .

Question 2 – Solution

Step 1: State the two-sided estimate.

For a positive decreasing function f with $a_n = f(n)$,

$$\int_{N+1}^{\infty} f(x) dx \leq R_N \leq \int_N^{\infty} f(x) dx.$$

Here $f(x) = x^{-2}$ is positive, continuous, and decreasing.

Step 2: Evaluate both integrals.

$$\int_{N+1}^{\infty} \frac{dx}{x^2} = \frac{1}{N+1}, \quad \int_N^{\infty} \frac{dx}{x^2} = \frac{1}{N}.$$

Thus

$$\boxed{\frac{1}{N+1} \leq R_N \leq \frac{1}{N}}.$$

Step 3: Convert the tail bracket to a sum bracket.

Since $S = S_N + R_N$,

$$\boxed{S_N + \frac{1}{N+1} \leq S \leq S_N + \frac{1}{N}}.$$

The bracket width is $1/[N(N+1)]$.