

Question 4

Analyze $\sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2 4^n}$.

- (a) Show why the Ratio Test is inconclusive.
- (b) Use a standard central-binomial-coefficient asymptotic to reach a conclusion.

Question 4 – Solution

Step 1: Try the natural factorial test.

If $a_n = (2n)!/[(n!)^2 4^n]$, then

$$\frac{a_{n+1}}{a_n} = \frac{(2n+2)(2n+1)}{4(n+1)^2} = \frac{2n+1}{2n+2} \rightarrow 1.$$

The Ratio Test is therefore inconclusive.

Step 2: Replace it with asymptotic comparison.

Stirling's formula yields the standard estimate

$$\binom{2n}{n} \sim \frac{4^n}{\sqrt{\pi n}}.$$

Consequently,

$$a_n = \frac{1}{4^n} \binom{2n}{n} \sim \frac{1}{\sqrt{\pi n}}.$$

Equivalently, $a_n/(1/\sqrt{n}) \rightarrow 1/\sqrt{\pi} > 0$.

Conclusion.

Since $\sum n^{-1/2}$ diverges, the given series **diverges** by Limit Comparison. A ratio limit of 1 signaled that a finer growth estimate was needed.