

Question 8

Analyze

$$\sum_{n=1}^{\infty} \left(\frac{2 + \sin n}{4} \right)^n.$$

Use the limsup form of the Root Test to handle the oscillating base, and explain why an ordinary root limit is unavailable.

Question 8 – Solution**Step 1: Take the root.**Since $1 \leq 2 + \sin n \leq 3$,

$$\sqrt[n]{|a_n|} = \frac{2 + \sin n}{4} \in \left[\frac{1}{4}, \frac{3}{4} \right].$$

This sequence oscillates rather than approaching one value.

Step 2: Compute the limsup.Integer angles modulo 2π are dense because $1/(2\pi)$ is irrational. Consequently, $\sin n$ comes arbitrarily close to 1, so

$$L = \limsup_{n \rightarrow \infty} \frac{2 + \sin n}{4} = \frac{3}{4} < 1.$$

(For convergence alone, the uniform bound $\sqrt[n]{|a_n|} \leq 3/4$ already suffices.)**Conclusion.**The series **converges absolutely** by the limsup Root Test.