

**Question 7**

Use the Root Test to determine whether

$$\sum_{n=2}^{\infty} \left(1 - \frac{1}{\sqrt{n}}\right)^{n^2}$$

converges. Use logarithms to evaluate the limit of the  $n$ th root.

**Question 7 – Solution**

**Step 1: Take the  $n$ th root.**

$$r_n = \sqrt[n]{a_n} = \left(1 - \frac{1}{\sqrt{n}}\right)^n.$$

**Step 2: Take logarithms.**

Since  $\log(1 - u) \leq -u$  for  $0 < u < 1$ ,

$$\log r_n = n \log \left(1 - \frac{1}{\sqrt{n}}\right) \leq -\sqrt{n} \longrightarrow -\infty.$$

Thus  $0 \leq r_n \leq e^{-\sqrt{n}} \rightarrow 0$ . (The expansion  $\log(1 - u) = -u - u^2/2 + O(u^3)$  gives the sharper relation  $\log r_n = -\sqrt{n} - 1/2 + O(n^{-1/2})$ .)

**Conclusion.**

The root limit is  $L = 0 < 1$ , so the series **converges absolutely** and does so faster than any fixed geometric rate.