

Question 10

Use the Ratio Test to analyze $\sum_{n=1}^{\infty} \left(\frac{n}{n+1} \right)^{n^2}$. Use logarithms and a standard logarithmic expansion to evaluate the ratio limit.

Question 10 – Solution

Step 1: Describe the term logarithmically.

Let $a_n = (n/(n+1))^{n^2}$. Then

$$\log a_n = -n^2 \log \left(1 + \frac{1}{n} \right).$$

Using $\log(1+t) = t - t^2/2 + O(t^3)$,

$$\log a_n = -n^2 \left(\frac{1}{n} - \frac{1}{2n^2} + O(n^{-3}) \right) = -n + \frac{1}{2} + O(n^{-1}).$$

Step 2: Find the ratio limit.

$$\log \left(\frac{a_{n+1}}{a_n} \right) = \log a_{n+1} - \log a_n = -1 + O(n^{-1}) \rightarrow -1.$$

Exponentiating gives $L = e^{-1} < 1$.

Conclusion.

The series **converges**. More specifically, $a_n \sim e^{1/2}e^{-n}$, so its terms have geometric-scale decay.